Miskolc Mathematical Notes
Vol. 12 (2011), No 1, pp. 61-73

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2011.278

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Božidar Ivanković, Josip Pečarić, and Sanja Varošanec

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NEW RESULTS ON THE AM-GM INEQUALITY

BOŽIDAR IVANKOVIĆ, JOSIP PEČARIĆ, AND SANJA VAROŠANEC

Received 11 May, 2010

Abstract. There is a large number of discrete inequalities between geometric and arithmetic means. Our intention is to investigate possible generalization and integral analogues of the inequalities obtained by Redhefer, Bullen and Godunova. At the end, we will consider refinements of these inequalities by different separations of their domains.

2000 *Mathematics Subject Classification:* 26D15

Keywords: convex functions, Jensen's inequality

1. PRELIMINARES

Consider the following sequences of positive real numbers:

$$(a)=(a_1,\dots,a_n);\ (b)=(b_1,\dots,b_n);\ (p)=(p_1,\dots,p_n);\ (q)=(q_1,\dots,q_n).$$

We use the notations:

$$P_n=\sum_{k=1}^n p_k, Q_n=\sum_{k=1}^n q_k,$$
$$A_n(a,q)=\frac{1}{Q_n}\sum_{k=1}^n q_k a_k, G_n(a,q)=\prod_{k=1}^n a_k^{\frac{q_k}{Q_n}}, \tag{1.1}$$
$$\Gamma_n(a,q)=\frac{1}{Q_n}\sum_{k=1}^n q_k G_k(a,q).$$

A function $F:[a,b]\to\mathbb{R}$ is said to be convex if for all $x,y\in[a,b]$ and all $\alpha\in[0,1]$,

$$F(\alpha x+(1-\alpha)y)\le\alpha F(x)+(1-\alpha)F(y). \tag{1.2}$$

If the inequality (1.2) is reversed, then F is said to be concave.

Investigating recurrent inequalities in [5], Redhefer has obtained the following result involving the arithmetic and geometric means. The statement is given in terms of (1.1), where $G_n(a,1)=G_n(a,q)$, $q_k=1$, $k=1,\dots,n$.

Theorem 1 (Redhefer)**.** *If $t > 0$, then*

$$G_n(a,1) < A_n(a,1)e^{-t} + t\Gamma_n(a,1).$$

Bullen [2] proved an inequality of Rado type generalizing Redheffer's inequality.

Theorem 2 (Bullen)**.** *If $0 \le tq_n \le Q_n$, then*

$$\begin{aligned} &Q_n(A_n(a,q)e^{-t} + t\Gamma_n(a,q) - G_n(a,q)) \\ \ge\ &Q_{n-1}(A_{n-1}(a,q)e^{-t} + t\Gamma_{n-1}(a,q) - G_{n-1}(a,q)). \end{aligned}$$

A more general result (given bellow) is obtained in [3].

Theorem 3 (Godunova)**.** *Suppose that G and F are twice continuously differentiable functions such that G is concave and $GF = (G \circ F)$ is convex.*

For $t \in \mathbb{R}$ the next inequality is true:

$$\begin{aligned} &Q_n\left\{G(A_n(F(t+a_k),q)) - tA_n\left((GF)'\left(\frac{q_kP_k}{p_kQ_k}A_k(a,p)\right),q\right)\right. && (1.3)\\ &\qquad\left. -GF\left(\frac{q_nP_n}{p_nQ_n}A_n(a,p)\right)\right\} && (1.4)\\ \ge\ &Q_{n-1}\left\{G(A_{n-1}(F(t+a_k),q)) - tA_{n-1}\left((GF)'\left(\frac{q_kP_k}{p_kQ_k}A_k(a,p)\right),q\right)\right. && (1.5)\\ &\qquad\left. -GF\left(\frac{q_nP_{n-1}}{p_nQ_{n-1}}A_{n-1}(a,p)\right)\right\}. && (1.6) \end{aligned}$$

In this article we will present some generalizations and refinements of these results.

2. Main result

Our results are based on the well-known properties of convex functions. The conditions are simpler than the ones of Theorem 3. It is obvious that Theorem 3 follows from this result.

Theorem 4. *If F is a convex differentiable function, then for all $t \in \mathbb{R}$*

$$\begin{aligned} &Q_n\left[A_n(F(t+a_k),q) - t\cdot A_n\left(F'\left(\frac{q_kP_k}{p_kQ_k}\cdot A_k(a,p)\right),q\right)\right. \\ &\qquad\left. -F\left(\frac{q_nP_n}{p_nQ_n}\cdot A_n(a,p)\right)\right] \\ \ge\ &Q_{n-1}\left[A_{n-1}(F(t+a_k),q) - t\cdot A_{n-1}\left(F'\left(\frac{q_kP_k}{p_kQ_k}\cdot A_k(a,p)\right),q\right)\right. \\ &\qquad\left. -F\left(\frac{q_nP_{n-1}}{p_nQ_{n-1}}\cdot A_{n-1}(a,p)\right)\right]. \end{aligned} \tag{2.1}$$

Proof. Obviously, we have

$$\sum_{k=1}^{n} q_k F(t+a_k) - \sum_{k=1}^{n-1} q_k F(t+a_k) = q_n F(t+a_n)$$

and

$$Q_n \cdot A_n\left(F'\left(\frac{q_k P_k}{p_k Q_k}\cdot A_k(a,p)\right),q\right) - Q_{n-1}\ cdot A_{n-1}\left(F'\left(\frac{q_k P_k}{p_k Q_k}\cdot A_k(a,p)\right),q\right) = q_n \cdot F'\left(\frac{q_n P_n}{p_n Q_n}\cdot A_n(a,p)\right).$$

Statement (2.1) is equivalent to

$$q_n F(t+a_n) + Q_{n-1}\cdot F\left(\frac{q_n P_{n-1}}{p_n Q_{n-1}}\cdot A_{n-1}(a,p)\right) \geq Q_n F\left(\frac{q_n P_n}{p_n Q_n}\cdot A_n(a,p)\right) + t q_n F'\left(\frac{q_n P_n}{p_n Q_n}\cdot A_n(a,p)\right), \tag{2.2}$$

so it remains to show (2.2). Because of $\frac{q_n}{Q_n} + \frac{Q_{n-1}}{Q_n} = 1$, the definition of convex functions implies

$$q_n \cdot F(t+a_n) + Q_{n-1}\cdot F\left(\frac{q_n P_{n-1}}{p_n Q_{n-1}}\cdot A_{n-1}(a,p)\right) \geq Q_n \cdot F\left(\frac{t q_n}{Q_n} + \frac{q_n P_n}{p_n Q_n}\cdot A_n(a,p)\right). \tag{2.3}$$

Presuming that the convex function F is differentiable, we have (see [4], p.5, Theorem 1.6.):

$$F(u+\alpha) \geq F(u) + \alpha \cdot F'(u). \tag{2.4}$$

Upon application of these results, the following inequality is obtained

$$Q_n \cdot F\left(\frac{q_n P_n}{p_n Q_n}\cdot A_n(a,p) + \frac{t q_n}{Q_n}\right) \geq Q_n \cdot F\left(\frac{q_n P_n}{p_n Q_n}\cdot A_n(a,p)\right) + t\cdot q_n \cdot F'\left(\frac{q_n P_n}{p_n Q_n}\cdot A_n(a,p)\right). \tag{2.5}$$

Now (2.3) and (2.5) imply (2.2). □

Corollary 1. *If F is a strictly monotone convex function, then*

$$Q_n\left[A_n(b,q)-F\left(\frac{q_nP_n}{p_nQ_n}A_n(F^{-1}(b),p)\right)\right]$$
$$\geq Q_{n-1}\left[A_{n-1}(b,q)-F\left(\frac{q_nP_{n-1}}{p_nQ_{n-1}}A_{n-1}(F^{-1}(b),p)\right)\right].$$

Proof. It is enough to substitute $t=0$ and $a_k=F^{-1}(b_k),\ k=1,2,\dots,n$ in (2.1). □

Corollary 2. *If F is a convex, differentiable function, then for every $t\in\mathbb{R}$*

$$F(A_n(a,p))\leq A_n(F(t+a_k),p)-tA_n(F'(A_k(a,p),p). \tag{2.6}$$

Proof. Substitute $(p)=(q)$ in (2.1) for $n=2,3,\dots,n-1,n$ in order to get a sequence of inequalities. Then sum these inequalities. □

Inequality (2.6) turns into the inequality of Redhefer upon taking $F(u)=e^{-u}$, $a_i=\ln\dfrac{1}{b_i}$ and $(p)=(1,\dots,1)$. This inequality is proved in [5] for $t\geq 0$. The same inequality is proved in [2], but only for $0\leq t\leq 2$.

Theorem 4 ensures the statement of Theorem 2 proved in [2] only for $0\leq tq_n\leq Q_n$.

Corollary 3. *For every $t\in\mathbb{R}$*

$$Q_n\left(A_n(b,q)e^{-t}+t\Gamma_n(b,q)-G_n(b,q)\right)$$
$$\geq Q_{n-1}\left(A_{n-1}(b,q)e^{-t}+t\Gamma_{n-1}(b,q)-G_{n-1}(b,q)\right). \tag{2.7}$$

Proof. To obtain (2.7), substitute $(p)=(q)$ and $F(u)=e^{-u}$ in (2.1). Then $F(a_k)=b_k=e^{-a_k}$ and $a_k=-\ln b_k=\ln\dfrac{1}{b_k}$. □

Remark. Theorem 3, (the first theorem of [3]), follows from Theorem 4. Substitute the composition GF instead of the convex function F in (2.2) and use the Jensen inequality for the concave function G:

$$Q_nG(A_n(F(t+a_k),q))\geq Q_{n-1}G(A_{n-1}(F(t+a_k),q))+q_nG(F(t+a_n)).$$

The next inequality from [1] is an immediate consequence of Theorem 3.

Corollary 4. *Suppose that F is a strictly monotone function, G is a concave function and GF is a convex function. Then*

$$Q_n\left\{G(A_n(b,q))-(GF)\left(\frac{q_nP_n}{p_nQ_n}A_n(F^{-1}(b),p)\right)\right\}$$
$$\geq Q_{n-1}\left\{G(A_{n-1}(b,q))-(GF)\left(\frac{q_nP_{n-1}}{p_nQ_{n-1}}A_{n-1}(F^{-1}(b),p)\right)\right\}.$$

Taking $(p) = (q)$, the next corollary is obtained from (2.1).

Corollary 5. *If F is a differentiable convex function, then*

$$\begin{aligned} &\sum_{k=1}^{n} p_k F(t+a_k) - t \cdot \sum_{k=1}^{n} p_k F'\left(\frac{1}{P_k}\sum_{j=1}^{k} p_j a_j\right) \\ &\qquad - \sum_{k=1}^{n} p_k \cdot F\left(\frac{1}{P_n}\sum_{k=1}^{n} p_k a_k\right) \\ &\geq \sum_{k=1}^{n-1} p_k F(t+a_k) - t \cdot \sum_{k=1}^{n-1} p_k F'\left(\frac{1}{P_k}\sum_{j=1}^{k} p_j a_j\right) \\ &\qquad - \sum_{k=1}^{n-1} p_k \cdot F\left(\frac{1}{P_{n-1}}\sum_{k=1}^{n-1} p_k a_k\right), \end{aligned}$$

whereby $t \in \mathbb{R}$.

3. An Application on Integral analogue

Integral analogue is given in [3] as Theorem 2.

Theorem 5 (Godunova)**.** *Let ψ be a convex, differentiable function and let $p(\tau)$ and $a(\tau)$ be positive functions for $\tau > 0$. Then the function*

$$\begin{aligned} \Phi(x) = \int_0^x \psi(t+a(\tau))p(\tau)d\tau - t\int_0^x \psi'\left(\frac{\int_0^\tau a(\xi)p(\xi)d\xi}{\int_0^\tau p(\xi)d\xi}\right)p(\tau)d\tau& \\ - \psi\left(\frac{\int_0^x a(\tau)p(\tau)d\tau}{\int_0^x p(\tau)d\tau}\right)\cdot \int_0^x p(\tau)d\tau& \end{aligned} \tag{3.1}$$

is an increasing function on $(0,\infty)$.

The next corollary estimates the difference between the values, so that composition GF is calculated at the mean values on the intervals $[0,x] \subset [0,y]$.

Corollary 6. *Let GF be a convex function, let G be a concave function and suppose that a, p are positive on $(0,\infty)$. Then*

$$\begin{aligned} &GF\left(\frac{\int_0^y a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau}\right)\cdot\int_0^y p(\tau)d\tau \\ -&GF\left(\frac{\int_0^x a(\tau)p(\tau)d\tau}{\int_0^x p(\tau)d\tau}\right)\cdot\int_0^x p(\tau)d\tau \end{aligned}$$

$$\le \int_x^y p(\tau)d\tau \cdot G\left(\frac{\int_x^y F(t+a(\tau))p(\tau)d\tau}{\int_x^y p(\tau)d\tau}\right)$$
$$-t\int_x^y (GF)'\left(\frac{\int_0^\tau a(\xi)p(\xi)d\xi}{\int_0^\tau p(\xi)d\xi}\right)p(\tau)d\tau.$$

Proof. Substitute $\psi = GF$ in (5). Notice that if
$\Phi(x) = \int_0^x GF(t+a(\tau))p(\tau)d\tau + D(x)$, then for $0 \le x \le y$ we have

$$\begin{aligned}
&D(x)-D(y)\\
=&t\int_0^y (GF)'\left(\frac{\int_0^\tau a(\xi)p(\xi)d\xi}{\int_0^\tau p(\xi)d\xi}\right)p(\tau)d\tau\\
&+GF\left(\frac{\int_0^y a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau}\right)\cdot\int_0^y p(\tau)d\tau\\
&-t\int_0^x (GF)'\left(\frac{\int_0^\tau a(\xi)p(\xi)d\xi}{\int_0^\tau p(\xi)d\xi}\right)p(\tau)d\tau\\
&-GF\left(\frac{\int_0^x a(\tau)p(\tau)d\tau}{\int_0^x p(\tau)d\tau}\right)\cdot\int_0^x p(\tau)d\tau\\
\le&\int_0^y GF(t+a(\tau))p(\tau)d\tau - \int_0^x GF(t+a(\tau))p(\tau)d\tau\\
=&\int_x^y GF(t+a(\tau))p(\tau)d\tau\\
\le&\int_x^y p(\tau)d\tau\cdot G\left(\frac{\int_x^y F(t+a(\tau))p(\tau)d\tau}{\int_x^y p(\tau)d\tau}\right).
\end{aligned}$$

□

In case that all calculations are well defined, Jensen's inequality for a concave function G

$$\int_0^y GF(t+a(\tau))p(\tau)d\tau \le P(y)\cdot G\left(\frac{1}{P(y)}\int_0^y F(t+a(\tau))p(\tau)d\tau\right),$$

with $P(y) = \int_0^y p(\tau)d\tau$, makes it possible to define the function

$$\mathcal{F}(y) = P(y)\cdot G\left(\frac{1}{P(y)}\int_0^y F(t+a(\tau))p(\tau)d\tau\right) - \int_0^y GF(t+a(\tau))p(\tau)d\tau.$$

The function $\mathcal{F}$ is monotone on $(0,\infty)$.

Proposition 1. *If G is a concave function, then $0 \le x \le y$ ensures*

$$\mathcal{F}(x) \le \mathcal{F}(y). \tag{3.2}$$

Proof. This statements follows from the concavity of G. There is a chain of inequalities:

$$\begin{aligned}
P(x)G\left(\frac{1}{P(x)}\int_0^x F(t+a(\tau))p(\tau)d\tau\right) - \int_0^x GF(t+a(\tau))p(\tau)d\tau& \\
+\int_0^y GF(t+a(\tau))p(\tau)d\tau& \\
= P(x)G\left(\frac{1}{P(x)}\int_0^x F(t+a(\tau))p(\tau)d\tau\right)& \\
+\frac{\int_x^y p(\tau)d\tau}{\int_x^y p(\tau)d\tau}\cdot\int_x^y GF(t+a(\tau))p(\tau)d\tau& \\
\le \frac{P(y)}{P(y)}\left[P(x)\cdot G\left(\frac{1}{P(x)}\int_0^x F(t+a(\tau))p(\tau)d\tau\right)\right.& \\
\left.+\int_x^y p(\tau)d\tau\cdot G\left(\frac{\int_x^y F(t+a(\tau))p(\tau)d\tau}{\int_x^y p(\tau)d\tau}\right)\right]& \\
= P(y)\left[\frac{P(x)}{P(y)}\cdot G\left(\frac{1}{P(x)}\int_0^x F(t+a(\tau))p(\tau)d\tau\right)\right.& \\
\left.+\frac{\int_x^y p(\tau)d\tau}{P(y)}\cdot G\left(\frac{\int_x^y F(t+a(\tau))p(\tau)d\tau}{\int_x^y p(\tau)d\tau}\right)\right]& \\
\le P(y)\cdot G\left(\frac{1}{P(y)}\int_0^x F(t+a(\tau))p(\tau)d\tau + \frac{1}{P(y)}\int_x^y F(t+a(\tau))p(\tau)d\tau\right)& \\
= P(y)\cdot G\left(\frac{1}{P(y)}\int_0^y F(t+a(\tau))p(\tau)d\tau\right).&
\end{aligned}$$

□

At this point observe certain refinements when G is concave and GF is a convex function.

Remark 1. Applying (3.2) in

$$D(x)-D(y) \le \int_0^y GF(t+a(\tau))p(\tau)d\tau - \int_0^x GF(t+a(\tau))p(\tau)d\tau,$$

the following is obtained:

$$P(y)\cdot GF\left(\frac{\int_0^y a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau}\right) - P(x)\cdot GF\left(\frac{\int_0^x a(\tau)p(\tau)d\tau}{\int_0^x p(\tau)d\tau}\right)$$

$$\leq P(y)\cdot G\left(\frac{\int_0^y F(t+a(\tau))p(\tau)d\tau}{\int_0^y p(\tau)d\tau}\right) - P(x)\cdot G\left(\frac{\int_0^x F(t+a(\tau))p(\tau)d\tau}{\int_0^x p(\tau)d\tau}\right)$$
$$-t\int_x^y (GF)'\left(\frac{\int_0^\tau a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau}\right)p(\tau)d\tau.$$

To sum up, if GF is a non-decreasing function, we can make the following observation.

Remark 2. If GF is a nondecreasing differentiable function, then $GF' \geq 0$ and for $t \geq 0$ one can obtain

$$P(y)\cdot GF\left(\frac{\int_0^y a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau}\right) - P(x)\cdot GF\left(\frac{\int_0^x a(\tau)p(\tau)d\tau}{\int_0^x p(\tau)d\tau}\right)$$
$$\leq P(y)\cdot G\left(\frac{\int_0^y F(t+a(\tau))p(\tau)d\tau}{\int_0^y p(\tau)d\tau}\right) - P(x)\cdot G\left(\frac{\int_0^x F(t+a(\tau))p(\tau)d\tau}{\int_0^x p(\tau)d\tau}\right).$$

For $t = 0$ we get ordinary monotonicity. If $[0,x] \subseteq [0,y]$, then

$$P(x)\cdot G\left(\frac{\int_0^x F(a(\tau))p(\tau)d\tau}{\int_0^x p(\tau)d\tau}\right) - P(x)\cdot GF\left(\frac{\int_0^x a(\tau)p(\tau)d\tau}{\int_0^x p(\tau)d\tau}\right)$$
$$\leq P(y)\cdot G\left(\frac{\int_0^y F(a(\tau))p(\tau)d\tau}{\int_0^y p(\tau)d\tau}\right) - P(y)\cdot GF\left(\frac{\int_0^y a(\tau)p(\tau)d\tau}{\int_0^y p(\tau)d\tau}\right).$$

4. DECREASING PROPERTY UPON TAKING SUBSETS

In the next theorem an integral version of the statement obtained in (1.6) is given whereby $(p) = (q)$ and $t = 0$. The statement appeared as particular case of Bullen's inequality (see [4]). In [3] the author omitted the proof. We are providing it here to refresh certain basic integral inequalities.

Theorem 6. *Let $\mathcal{D}$ be a measurable domain and let $\mathcal{D}_1 \subset \mathcal{D}$ be its regular measurable sub-domain. Further, suppose $a,q : \mathcal{D} \to [0,+\infty)$ is such that $Q = \int_{\mathcal{D}} q(u)dV_u \neq 0$ and $Q_1 = \int_{\mathcal{D}_1} q(u)dV_u \neq 0$. If G is concave, F is monotone and GF is a convex function, then*

$$Q_1\left[G\left(\frac{1}{Q_1}\int_{\mathcal{D}_1} F(a(u))q(u)dV_u\right) - GF\left(\frac{1}{Q_1}\int_{\mathcal{D}_1} a(u)q(u)dV_u\right)\right]$$
$$\leq Q\left[G\left(\frac{1}{Q}\int_{\mathcal{D}} F(a(u))q(u)dV_u\right) - GF\left(\frac{1}{Q}\int_{\mathcal{D}} a(u)q(u)dV_u\right)\right]. \tag{4.1}$$

Proof. Concavity of G ensures

$$\begin{aligned} &Q\cdot G\left(\frac{1}{Q}\int_{\mathcal{D}}F(a(u))q(u)dV_u\right)\\ \geq &Q_1\cdot G\left(\frac{1}{Q_1}\int_{\mathcal{D}_1}F(a(u))q(u)dV_u\right)\\ &+Q_2\cdot G\left(\frac{1}{Q_2}\int_{\mathcal{D}_2}F(a(u))q(u)dV_u\right),\end{aligned}$$

wherby $\mathcal{D}_2=\mathcal{D}\setminus\mathcal{D}_1$ with $Q_2=\int_{\mathcal{D}_2}q(u)dV_u\neq 0$. Besides, the convexity of GF implies

$$\begin{aligned} &Q\cdot GF\left(\frac{1}{Q}\int_{\mathcal{D}}a(u)q(u)dV_u\right)\\ &\leq Q_1\cdot GF\left(\frac{1}{Q_1}\int_{\mathcal{D}_1}a(u)q(u)dV_u\right)+Q_2\cdot GF\left(\frac{1}{Q_2}\int_{\mathcal{D}_2}a(u)q(u)dV_u\right).\end{aligned}$$

Integral version of Jensen's inequality (see [4], p. 45) for concave function G provides

$$Q_2\cdot G\left(\frac{1}{Q_2}\int_{\mathcal{D}_2}F(a(u))q(u)dV_u\right)\geq\int_{\mathcal{D}_2}GF(a(u))q(u)dV_u.$$

Furthermore, the convexity of GF provides the statement

$$\int_{\mathcal{D}_2}GF(a(u))q(u)dV_u\geq Q_2\cdot GF\left(\frac{1}{Q_2}\int_{\mathcal{D}_2}a(u)q(u)dV_u\right).$$

Now there is a chain

$$\begin{aligned} &Q\cdot G\left(\frac{1}{Q}\int_{\mathcal{D}}F(a(u))q(u)dV_u\right)-Q_1\cdot G\left(\frac{1}{Q_1}\int_{\mathcal{D}_1}F(a(u))q(u)dV_u\right)\\ &=Q_2\cdot G\left(\frac{1}{Q_2}\int_{\mathcal{D}_2}F(a(u))q(u)dV_u\right)\geq Q_2\cdot GF\left(\frac{1}{Q_2}\int_{\mathcal{D}_2}a(u)q(u)dV_u\right)\\ &\geq Q\cdot GF\left(\frac{1}{Q}\int_{\mathcal{D}}a(u)q(u)dV_u\right)-Q_1\cdot GF\left(\frac{1}{Q_1}\int_{\mathcal{D}_1}a(u)q(u)dV_u\right).\end{aligned}$$

The first line and the last line from this give (4.1). □

Finally, we generalize (4.1).

Theorem 7. *Let (Ω,Σ,μ) be a space with a positive finite measure and let $a:\Omega\to\mathbb{R}$ be a μ-measurable function. Suppose that G and F are differentiable functions such that G is concave and GF is convex. For every $t\in\mathbb{R}$, if $\Omega_1\in\Sigma$, and $\Omega_1\neq\Omega$, then*

$$\mu(\Omega)\cdot G\left(\frac{1}{\mu(\Omega)}\int_{\Omega}F(t+a)d\mu\right)\tag{4.2}$$

$$-\mu(\Omega_1)\cdot G\left(\frac{1}{\mu(\Omega_1)}\int_{\Omega_1} F(t+a)d\mu\right)$$
$$\geq (\mu(\Omega)-\mu(\Omega_1)\left[GF\left(\frac{\int_\Omega a\,d\mu-\int_{\Omega_1} a\,d\mu}{\mu(\Omega)-\mu(\Omega_1)}\right)\right.$$
$$\left.+t\cdot(GF)'\left(\frac{\int_\Omega a\,d\mu-\int_{\Omega_1} a\,d\mu}{\mu(\Omega)-\mu(\Omega_1)}\right)\right].$$

Proof. Take $\Omega\setminus\Omega_1=\Omega_2$. Then concavity of G ensures

$$\mu(\Omega)\cdot G\left(\frac{1}{\mu(\Omega)}\int_{\Omega} F(t+a)d\mu\right)-\mu(\Omega_1)\cdot G\left(\frac{1}{\mu(\Omega_1)}\int_{\Omega_1} F(t+a)d\mu\right)$$
$$\geq \mu(\Omega_2)\cdot G\left(\frac{1}{\mu(\Omega_2)}\int_{\Omega_2} F(t+a)d\mu\right). \quad (4.3)$$

Concavity of G implies (see [4], p.51) that

$$\mu(\Omega_2)\cdot G\left(\frac{1}{\mu(\Omega_2)}\int_{\Omega_2} F(t+a)d\mu\right)\geq \int_{\Omega_2} GF(t+a)d\mu.$$

Now, convexity of GF implies

$$\int_{\Omega_2} GF(t+a)d\mu\geq \mu(\Omega_2)\cdot GF\left(\frac{1}{\mu(\Omega_2)}\int_{\Omega_2}(t+a)d\mu\right).$$

Using (2.4) and the equality $\frac{1}{\mu(\Omega_2)}\int_{\Omega_2}(t+a)d\mu=t+\frac{1}{\mu(\Omega_2)}\int_{\Omega_2} a\,d\mu$, we claim that

$$GF\left(t+\frac{1}{\mu(\Omega_2)}\int_{\Omega_2} a\,d\mu\right)\geq GF\left(\frac{1}{\mu(\Omega_2)}\int_{\Omega_2} a\,d\mu\right)$$
$$+t(GF)'\left(\frac{1}{\mu(\Omega_2)}\int_{\Omega_2} a\,d\mu\right). \quad (4.4)$$

Finally, (4.3) and (4.4), together with the well-known properties $\mu(\Omega\setminus\Omega_1)=\mu(\Omega)-\mu(\Omega_1)$ and $\int_{\Omega\setminus\Omega_1} a\,d\mu=\int_\Omega a\,d\mu-\int_{\Omega_1} a\,d\mu$ give (4.2). □

Theorem 7 contains (4.1) as a particular case.

Remark 3. Notice that convexity of GF, assumed in (4.4), implies

$$\mu(\Omega)\cdot G\left(\frac{1}{\mu(\Omega)}\int_{\Omega} F(t+a)d\mu\right)-\mu(\Omega_1)\cdot G\left(\frac{1}{\mu(\Omega_1)}\int_{\Omega_1} F(t+a)d\mu\right)$$
$$\geq \mu(\Omega)\cdot GF\left(\frac{1}{\mu(\Omega)}\int_{\Omega} a\,d\mu\right)-\mu(\Omega_1)\cdot GF\left(\frac{1}{\mu(\Omega_1)}\int_{\Omega_1} a\,d\mu\right)$$

$$+t\cdot(GF)'\left(\frac{\int_{\Omega} a\, d\mu-\int_{\Omega_1} a\, d\mu}{\mu(\Omega)-\mu(\Omega_1)}\right) \tag{4.5}$$

and (4.1) can be obtained from (4.5) for $t=0$.

In the similar manner as in [6], where the authors have refined Jensen's inequality, here we give an estimate after dividing the n-tuples $(a_1,\dots,a_n)$, $(p_1,\dots,p_n)$ and $(q_1,\dots,q_n)$ into groups. Firstly, we obtain the next Theorem.

Theorem 8. *Suppose that G and F are differentiable functions such that G is concave and GF is convex. If $m\le n$, then*

$$\begin{aligned}
&Q_n\left[G(A_n(F(t+a_k),q))-tA_n\left((GF)'\left(\frac{q_kP_k}{p_kQ_k}A_k(a,p)\right),q\right)\right.\\
&\qquad\left.-GF\left(\frac{q_nP_n}{p_nQ_n}A_n(a,p)\right)\right]\\
&-Q_m\left[G(A_m(F(t+a_k),q))-tA_m\left((GF)'\left(\frac{q_kP_k}{p_kQ_k}A_k(a,p)\right),q\right)\right.\\
&\qquad\left.-GF\left(\frac{q_mP_m}{p_mQ_m}A_m(a,p)\right)\right]\\
&\ge\sum_{k=m}^{n-1}Q_k\left[GF\left(\frac{q_kP_k}{p_kQ_k}A_k(a,p)\right)-\left(\frac{q_{k+1}P_k}{p_{k+1}Q_k}A_k(a,p)\right)\right].
\end{aligned}$$

Proof. No generality is lost if we separate each of n-tuples $(a_1,\dots,a_n)$, $(p_1,\dots,p_n)$ and $(q_1,\dots,q_n)$ by the choosing the first m members in the first group. For example: $(a_1,\dots,a_m,a_{m+1},\dots,a_n)$.

Then the first inequality is

$$\begin{aligned}
&Q_n\left[G(A_n(F(t+a_k),q))-tA_n\left((GF)'\left(\frac{q_kP_k}{p_kQ_k}A_k(a,p)\right),q\right)\right.\\
&\qquad\left.-GF\left(\frac{q_nP_n}{p_nQ_n}A_n(a,p)\right)\right]\\
&\ge Q_{n-1}\left[G(A_{n-1}(F(t+a_k),q))-tA_{n-1}\left((GF)'\left(\frac{q_kP_k}{p_kQ_k}A_k(a,p)\right),q\right)\right.\\
&\qquad\left.-GF\left(\frac{q_{n-1}P_{n-1}}{p_{n-1}Q_{n-1}}A_{n-1}(a,p)\right)\right]\\
&+Q_{n-1}\left[GF\left(\frac{q_{n-1}P_{n-1}}{p_{n-1}Q_{n-1}}A_{n-1}(a,p)\right)-GF\left(\frac{q_nP_{n-1}}{p_nQ_{n-1}}A_{n-1}(a,p)\right)\right].
\end{aligned}$$

The second one is

$$Q_{n-1}\left[G(A_{n-1}(F(t+a_k),q))-tA_{n-1}\left((GF)'\left(\frac{q_kP_k}{p_kQ_k}A_k(a,p)\right),q\right)\right.$$

$$
\begin{aligned}
&\qquad -GF\left(\frac{q_{n-1}P_{n-1}}{p_{n-1}Q_{n-1}}A_{n-1}(a,p)\right)\Bigg]\\
&\geq Q_{n-2}\left[G(A_{n-2}(F(t+a_k),q))-tA_{n-2}\left((GF)'\left(\frac{q_kP_k}{p_kQ_k}A_k(a,p)\right),q\right)\right.\\
&\qquad \left.-GF\left(\frac{q_{n-2}P_{n-2}}{p_{n-2}Q_{n-2}}A_{n-2}(a,p)\right)\right]\\
&+Q_{n-2}\left[GF\left(\frac{q_{n-2}P_{n-2}}{p_{n-2}Q_{n-2}}A_{n-2}(a,p)\right)-GF\%left(\frac{q_{n-1}P_{n-2}}{p_{n-1}Q_{n-2}}A_{n-2}(a,p)\right)\right.
\end{aligned}
$$

And finally, the last one is

$$
\begin{aligned}
&Q_{m+1}\left[G(A_{m+1}(F(t+a_k),q))-tA_{m+1}\left((GF)'\left(\frac{q_kP_k}{p_kQ_k}A_k(a,p)\right),q\right)\right.\\
&\qquad \left.-GF\left(\frac{q_{m+1}P_{m+1}}{p_{m+1}Q_{m+1}}A_{m+1}(a,p)\right)\right]\\
&\geq Q_m\left[G(A_m(F(t+a_k),q))-tA_m\left((GF)'\left(\frac{q_kP_k}{p_kQ_k}A_k(a,p)\right),q\right)\right].\\
&\qquad -GF\left(\frac{q_mP_m}{p_mQ_m}A_m(a,p)\right)\Bigg]\\
&\qquad +Q_m\left[GF\left(\frac{q_mP_m}{p_mQ_m}A_m(a,p)\right)-GF\left(\frac{q_{m+1}P_m}{p_{m+1}Q_m}A_m(a,p)\right)\right].
\end{aligned}
$$

The desired inequality in the statement is obtained by summing these inequalities. □

The sum $\sum_{k=m}^{n-1} Q_k\left[GF\left(\frac{q_kP_k}{p_kQ_k}A_k(a,p)\right)-\left(\frac{q_{k+1}P_k}{p_{k+1}Q_k}A_k(a,p)\right)\right]$ depends on the positive weights (p) and (q). Therefore it could be positive, negative or zero. When $(p)=(q)$, the sum is zero and and we obtain the following Corollary.

Corollary 7. *Suppose G and F fulfill the assumptions of Theorem 8. If $n\geq m$, then*

$$
\begin{aligned}
&Q_n\left[G(A_n(F(t+a_k),p))-tA_n\left((GF)'(A_k(a,p)),p\right)-GF\left(A_n(a,p)\right)\right]\\
&\geq Q_m\left[G(A_m(F(t+a_k),p))-tA_m\left((GF)'(A_k(a,p)),p\right)-GF\left(A_m(a,p)\right)\right].
\end{aligned}
$$

Finally, we state the following Corollary in the case of disjunct separation of the n-tuples as: $(a_1,\ldots,a_n)=(a_1,\ldots,a_m,a_{m+1},\ldots,a_n)$.

Corollary 8. *Suppose that $G(u)$ and $F(u)$ satisfy conditions given in Theorem 8. If $l+m=n$ and $(b_1,\ldots,b_l,c_1,\ldots,c_m)=(a_1,\ldots,a_n)$, then*

$$
Q_l\left\{G(A_l(F(t+b_k),p))-tA_l\left((GF)'(A_k(b,p)),p\right)\right.
$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad -GF\left(A_l(b,p)\right)\} \\
&+Q_m\left\{G(A_m(F(t+c_k),q)) - tA_m\left((GF)'\left(A_k(c,p)\right),p\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. -GF\left(A_m(c,p)\right)\right\} \\
&\le 2\cdot Q_n\left\{G(A_n(F(t+a_k),p)) - tA_n\left((GF)'\left(A_k(a,p)\right),p\right)\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left. -GF\left(A_n(a,p)\right)\right\}.
\end{aligned}
$$

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Authors’ addresses

Božidar Ivanković
Faculty of Transport and Trafic Engineering, University of Zagreb, Vukelićeva 4, 10000, Croatia
E-mail address: `bozidar.ivankovic@zg.t-com.hr`

Josip Pečarić
Faculty of Textile, University of Zagreb, Prilaz Baruna Filipovića 30, 10000 Zagreb, Croatia
E-mail address: `pecaric@element.hr`

Sanja Varošanec
Department of Mathematics, University of Zagreb, Croatia
E-mail address: `varosans@math.hr`